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The Converse of the Dominated Ergodic Theorem

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Let $\{T_t: -\infty < t < \infty\}$ be a one-parameter group of measure-preserving transformations on a probability space $(X, \mathscr{B}, \mu)$. The *maximal function* f^* of a function $f \in L^1(X, \mathscr{B}, \mu)$ is defined by

$$f^*(x) = \sup_{t>0} \frac{1}{t} \int_0^t |f(T_s x)| \, ds.$$

Let $\log^+ u = \max\{0, \log u\}$. According to Wiener's Dominated Ergodic Theorem [11] (which is analogous to the basic inequalities concerning the Hardy–Littlewood maximal function [4]), if $f \in L^p(X, \mathscr{B}, \mu)$ for some $p > 1$, then also $f^* \in L^p(X, \mathscr{B}, \mu)$; while if f is in the Zygmund class $L \log L$, which consists of all $g \in L^1(X, \mathscr{B}, \mu)$ for which

$$\int_X |g| \log^+ |g| \, d\mu < \infty$$

(see [12]), then $f^* \in L^1(X, \mathscr{B}, \mu)$.

In 1962 Burkholder [1] proved that if $X_1, X_2, \ldots$ is a sequence of independent identically-distributed random variables, then

$$X^* = \sup_n \frac{|X_1 + X_2 + \cdots + X_n|}{n}$$

is integrable if and only if each X_k is in $L \log L$. After Stein [10] had observed that the converse of the $L \log L$ result concerning the Hardy–Littlewood maximal function is also true, and Gundy [3] had investigated the matter for certain martingales, Ornstein [7] proved the converse of the Dominated Ergodic Theorem for the case of a single ergodic measure-preserving transformation. More recently, this question has also been considered by Derriennic [2] and Jones [6].

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In order to prove the converse of Wiener's Dominated Ergodic Theorem for an ergodic one-parameter flow (Theorem 2), we will, as in the proofs of Stein, Gundy, and Ornstein, find a reverse maximal inequality (Theorem 1), which can be integrated to yield the desired result. Our method extends Hartman's proof [5] of the Pointwise Ergodic Theorem, which is based on the Rising Sun Lemma of F. Riesz [8], [9] (Lemma 1). Jones (personal communication) has remarked that Theorem 2 follows also from the reverse maximal inequality of [10]; this approach also works in the case of an ergodic action of $\mathbb{R}^n$.[1]

LEMMA 1. *Let h be a real-valued continuous function defined on an interval $[a, b]$, and let S denote the set of those $t \in (a, b)$ for which there is t' with $a < t' < t$ such that $h(t') > h(t)$. Then S is open; if $S = \cup (a_k, b_k)$ is its decomposition as a union of disjoint open intervals, then $h(a_k) \geqslant h(b_k)$ for all k; and in fact equality holds except possibly when $b_k = b$.*

The following simple but clever trick also plays a central role in Hartman's proof:

LEMMA 2. *Let $\{E_t: t > 0\}$ be an increasing family of measurable subsets of X, let $E = \bigcup_t E_t$, and suppose that $\phi \in L^1(X, \mathscr{B}, \mu)$. Then*

$$\lim_{n\to\infty} \frac{1}{n} \int_0^n \int_{E_t} \phi(x)\, d\mu(x)\, dt = \int_E \phi(x)\, d\mu(x).$$

Proof. Denote Lebesgue measure on $\mathbb{R}$ by m. Since

$$(1/n)\, m\{t \in [0, n]: x \in E_t\} \to \chi_E(x),$$

the result follows immediately from Fubini's Theorem and the Dominated Convergence Theorem.

The left-hand inequality in the following Theorem is the Maximal Ergodic Theorem, which does not depend on the assumption that $\{T_t\}$ be ergodic and $\alpha > \int f\, d\mu$.

THEOREM 1. *Let $\{T_t: -\infty < t < \infty\}$ be an ergodic one-parameter group of measure-preserving transformations on a probability space $(X, \mathscr{B}, \mu)$, let $f \geqslant 0$ be in $L^1(X, \mathscr{B}, \mu)$, and suppose that $\alpha > \int f\, d\mu$. Then*

$$\alpha\mu\{x: f^*(x) > \alpha\} \leqslant \int_{\{x: f^*(x) > \alpha\}} f\, d\mu \leqslant 4\alpha\mu\{x: f^*(x) > \alpha\}.$$

Proof. For each $t > 0$, let

$$f_t^*(x) = \sup_{0 < \tau \leqslant t} \frac{1}{\tau} \int_0^\tau f(T_s x)\, ds;$$

[1] In joint work with Brian Marcus (to appear elsewhere), we have been able to show that the constant 4 in Theorem 1 can be replaced by 1, so that actually the left-hand inequality in Theorem 1 is an equality.

let $E = \{x: f^*(x) > \alpha\}$, $E_t = \{x: f_t^*(x) > \alpha\}$, $F_t = T_t E_t$, and $A_x = \{t > 0: x \in F_t\}$. Fix $n > 0$ and let $A_{x,n} = A_x \cap (0, n)$. Let

$$g(t) = \int_0^t f(T_{-s}x)\, ds$$

and $h(t) = \alpha t - g(t)$. Then $A_x = \{t > 0$: there is t' with $0 < t' < t$ such that $h(t') > h(t)\}$. By Lemma 1, $A_{x,n}$ is the union of open intervals (a_k, b_k), and

$$g(b_k) - g(a_k) \geqslant \alpha(b_k - a_k) \tag{1}$$

for all k, with equality holding except possibly when $b_k = n \in A_x$. It is certainly true, then, that

$$\begin{aligned} \int_{A_{x,n}} f(T_{-s}x)\, ds &= \sum_k \int_{a_k}^{b_k} f(T_{-s}x)\, ds = \sum_k [g(b_k) - g(a_k)] \\ &\geqslant \alpha \sum_k (b_k - a_k) = \alpha m(A_{x,n}). \end{aligned} \tag{2}$$

To deal with the possible inequality in case $n \in A_x$, let

$$\begin{aligned} \nu(x) &= 2n &&\text{if} \quad [n, 2n] \subset A_x \\ &= \inf([n, 2n] \backslash A_x) &&\text{otherwise} \end{aligned}$$

and $U_n = \{x: [n, 2n] \subset A_x\}$. If $x \in X \backslash U_n$, then when Lemma 1 is applied to $A_{x,\nu(x)}$ equality will hold in (1) for *every* open subinterval; therefore

$$\int_{X \backslash U_n} \int_{A_{x,\nu(x)}} f(T_{-s}x)\, ds\, d\mu = \alpha \int_{X \backslash U_n} m(A_{x,\nu(x)})\, d\mu(x). \tag{3}$$

When $x \in U_n$, let

$$D_{x,n} = \bigcup_{b_k \neq n} (a_k, b_k)$$

be the union of all the subintervals comprising $A_{x,n}$ except the rightmost one, (a_j, b_j). Then

$$\begin{aligned} \int_{U_n} \int_{A_{x,\nu(x)}} f(T_{-s}x)\, ds\, d\mu &= \int_{U_n} \left[\alpha \sum_{k \neq j} (b_k - a_k) + \int_{a_j}^{\nu(x)} f(T_{-s}x)\, ds\right] d\mu \\ &= \int_{U_n} \left[\alpha m(D_{x,n}) + \frac{2n - a_j}{2n - a_j} \int_{a_j}^{2n} f(T_{-s}x)\, ds\right] d\mu \\ &\leqslant \int_{U_n} \left[\alpha m(D_{x,n}) + (2n - a_j) \frac{2}{2n} \int_0^{2n} f(T_{-s}x)\, ds\right] d\mu \\ &\leqslant \int_{U_n} \max \left\{\alpha, 2 \frac{1}{2n} \int_0^{2n} f(T_{-s}x)\, ds\right\} m(A_{x,2n})\, d\mu. \end{aligned}$$

Now let $K_n = \{x \in U_n: 1/2n \int_0^{2n} f(T_{-s}x)\, ds \geqslant \alpha\}$. Since $\{T_t: -\infty < t < \infty\}$ is ergodic,

$$\lim_{n\to\infty} \frac{1}{2n} \int_0^{2n} f(T_{-s}x)\, ds = \int f\, d\mu < \alpha \qquad \text{a.e.},$$

so that $\mu(K_n) \to 0$. Because the measure $\int_G f\, d\mu$ is absolutely continuous with respect to μ, and $\{T_t\}$ preserves μ, $\int_{K_n} f(T_{-s}x)\, d\mu \to 0$ uniformly in s. Therefore

$$\int_{K_n} \max \left\{\alpha, 2\frac{1}{2n} \int_0^{2n} f(T_{-s}x)\, ds\right\} m(A_{x,2n})\, d\mu$$

$$= 2 \int_{K_n} \frac{1}{2n} \int_0^{2n} f(T_{-s}x)\, ds\; m(A_{x,2n})\, d\mu$$

$$\leqslant 4n \frac{1}{2n} \int_0^{2n} \int_{K_n} f(T_{-s}x)\, d\mu\, ds = o(n),$$

and hence

$$\frac{1}{n} \int_{U_n} \int_{A_{x,\nu(x)}} f(T_{-s}x)\, ds\, d\mu \leqslant \frac{1}{n} \int_{U_n \setminus K_n} 2\alpha m(A_{x,2n})\, d\mu + \epsilon_n,$$

where

$$\lim_{n\to\infty} \epsilon_n = 0.$$

Combining this with (3) gives

$$\frac{1}{n} \int_X \int_{A_{x,\nu(x)}} f(T_{-s}x)\, ds\, d\mu \leqslant 2\alpha \frac{1}{n} \int_X m(A_{x,2n})\, d\mu + \epsilon_n$$

$$= 2\alpha \frac{1}{n} \int_X \int_{A_{x,2n}} ds\, d\mu + \epsilon_n = 4\alpha \frac{1}{2n} \int_0^{2n} \int_{F_s} d\mu\, ds + \epsilon_n$$

$$= 4\alpha \frac{1}{2n} \int_0^{2n} \int_{E_s} d\mu\, ds + \epsilon_n;$$

similarly,

$$\frac{1}{n} \int_X \int_{A_{x,n}} f(T_{-s}x)\, ds\, d\mu = \frac{1}{n} \int_0^n \int_{F_s} f(T_{-s}x)\, d\mu\, ds$$

$$= \frac{1}{n} \int_0^n \int_{E_s} f\, d\mu\, ds.$$

Recalling (2),

$$\alpha \frac{1}{n} \int_0^n \int_{E_s} d\mu \, ds \leqslant \frac{1}{n} \int_0^n \int_{E_s} f \, d\mu \, ds \leqslant 4\alpha \frac{1}{2n} \int_0^{2n} \int_{E_s} d\mu \, ds + \epsilon_n .$$

The result then follows from Lemma 2 upon letting $n \to \infty$.

For each $\sigma \geqslant 0$, let $L \log^\sigma L$ denote the collection of all those measurable functions g on X for which

$$\int_X | g | (\log^+ | g |)^\sigma \, d\mu < \infty.$$

THEOREM 2. *Let $\{T_t\}$ be as in Theorem 1, let $f \in L^1(X, \mathscr{B}, \mu)$, and let $\sigma \geqslant 0$. Then $f^* \in L \log^\sigma L$ if and only if $f \in L \log^{\sigma+1} L$.*

Proof. Suppose without loss of generality that $f \geqslant 0$. For each $\alpha \geqslant 0$ let $G(\alpha) = \mu\{x: f^*(x) > \alpha\}$. Then a change of variables and integration by parts shows that

$$\int_X f^* \log^{+\sigma} f^* \, d\mu = - \int_0^\infty y \log^{+\sigma} y \, dG(y) = \int_0^\infty G(y) \, (\log^{+\sigma} y + \sigma \log^{+\sigma-1} y) \, dy$$

$$\geqslant K_{\sigma,f} \Big[1 + \int_{\|f\|_1}^\infty G(y) \log^{+\sigma} y \, dy \Big]$$

$$\geqslant K_{\sigma,f} \Big[1 + \int_{\|f\|_1}^\infty \log^{+\sigma} y \left(\frac{1}{4y} \int_{\{f^*>y\}} f \, d\mu \right) dy \Big]$$

$$\geqslant K_{\sigma,f} \Big[1 + \int_{\|f\|_1}^\infty \log^{+\sigma} y \left(\frac{1}{4y} \int_{\{f>y\}} f \, d\mu \right) dy \Big]$$

$$= K_{\sigma,f} \Big[1 + \int_{\{f>\|f\|_1\}} f(x) \int_{\|f\|_1}^{f(x)} \frac{\log^{+\sigma} y}{4y} \, dy \, d\mu(x) \Big]$$

$$= K'_{\sigma,f} \Big[1 + \frac{1}{4(\sigma+1)} \int_{\{f>\|f\|_1\}} f(x) \log^{+\sigma+1} f(x) \, d\mu(x) \Big],$$

for some constants $K_{\sigma,f}$ and $K'_{\sigma,f}$. Therefore $f \in L \log^{\sigma+1} L$ if $f^* \in L \log^\sigma L$.

For the converse we use an observation of Wiener's [11]: if Theorem 1 is applied to $g = f\chi_{\{f>\alpha/2\}}$, then $f^* \leqslant g^* + \alpha/2$ and

$$\mu\{f^* > \alpha\} \leqslant \mu \left\{ g^* > \frac{\alpha}{2} \right\} \leqslant \frac{2}{\alpha} \int_{\{g^*>\alpha/2\}} g \, d\mu \leqslant \frac{2}{\alpha} \int_X g \, d\mu = \frac{2}{\alpha} \int_{\{f>\alpha/2\}} f \, d\mu.$$

Therefore

$$\begin{aligned}\int_X f^* \log^{+\sigma} f^* \, d\mu &\leqslant M_{\sigma,f}\left[1 + \int_{2\|f\|_1}^{\infty} \log^{+\sigma} y\mu\{f^* > y\}\, dy\right] \\ &\leqslant M_{\sigma,f}\left[1 + \int_{2\|f\|_1}^{\infty} \log^{+\sigma} y \left(\frac{2}{y}\int_{\{f>y/2\}} f\, d\mu\right) dy\right] \\ &= M_{\sigma,f}\left[1 + \int_{\{f>\|f\|_1\}} f(x) \int_{\|f\|_1}^{f(x)} \frac{\log^{+\sigma} y}{y}\, dy\, d\mu\right] \\ &\leqslant M'_{\sigma,f}\left[1 + \int_{\{f>\|f\|_1\}} f \log^{+\sigma+1} f\, d\mu\right]\end{aligned}$$

for some constants $M_{\sigma,f}$ and $M'_{\sigma,f}$. This shows that if $f \in L \log^{\sigma+1} L$, then $f^* \in L \log^{\sigma} L$.

Among the interesting questions suggested by this circle of ideas, we mention the following two: (1) Are there analogues of Theorems 1 and 2 for "ergodic" positive operators T on $L^1(X, \mathscr{B}, \mu)$? (2) What are the properties of the (sublinear) operator $f \to f^*$ on the invariant subspace $\bigcap_{\sigma \geqslant 0} L \log^{\sigma} L$?

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